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Renormalisation group calculation of the eddy viscosity for isotropic turbulence

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Abstract. Iterative averaging, which has previously been shown to be equivalent to the renormalisation group, was applied directly to the Navier-Stokes equations in k space. The better frequency separation achieved by this technique (when compared to convention RG filtering and averaging operations) was shown to be due to the order in which operations were carried out, rather than to any difference in underlying assumptions. An expression for the eddy viscosity for wavenumber scales $k \leqslant k_c$ (say) was obtained in terms of the energy spectrum for $k \geqslant k_c$. For $k \ll k_c$, the eddy viscosity became constant. For $k \to k_c$ the eddy viscosity had a gentle roll-off, which may be compared with the cusp at $k = k_c$ found with renormalised perturbation theories.

1. Introduction

Renormalisation group (RG) methods (Wilson 1975) offer a powerful new approach to the problems of turbulence theory. This applies both to the laminar-turbulent transition and to the statistical description of well-developed turbulence, although it is only the latter which concerns us here.

As yet the application of RG to the statistical theory of turbulence is in its infancy and the supremacy of the renormalised perturbation theories (e.g. Kraichnan 1959, Kraichnan and Herring 1978, McComb 1978, McComb and Shanmugasundaram 1984a) in tackling the closure of the moment hierarchy is unlikely to be challenged for some time. A general discussion of RG and turbulence has been given by Kraichnan (1982). Here we shall restrict our attention to the relatively simple problem of calculating the mean effect of the small-scale motions (eddies) on the large scales. This offers a prescription for turbulent calculations, in that the large scales may be computed by direct numerical simulation, and the mean effect of the small scales represented analytically by an eddy viscosity. This technique (known as large eddy simulation) is currently receiving much attention in engineering studies of turbulence, with the subgrid-scale eddy viscosity being treated phenomenologically (e.g. see Voke and Collins 1983, Rogallo and Moin 1984).

Of course, the idea that eddies of wavenumber $k \geqslant k_c$ (say) act, on average, like an effective viscosity on eddies of $k \leqslant k_c$ is not new. Heisenberg (1948) used this idea as the basis of what is one of the better known phenomenological theories of the turbulent cascade. The basic idea has intuitive appeal, but, as Batchelor (1971) has pointed out, in the neighbourhood of $k = k_c$ it becomes difficult to justify. A quantitative analysis of this point has been given by Kraichnan (1976), who used renormalised

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perturbation theory to calculate the eddy viscosity $\nu(k|k_c)$ for $k \leqslant k_c$ in terms of the energy spectrum at $k \geqslant k_c$. Kraichnan found that $\nu(k|k_c) \to \text{constant} \times \alpha^{1/2}\varepsilon^{1/3}k_c^{-4/3}$ as $k \to \ll k_c$. This was based on the assumption that the energy spectrum was given by the Kolmogorov form

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3} \tag{1.1}$$

where α is constant and ε is the energy dissipation rate. With the same underlying assumption, the Heisenberg analysis gives a similar result. However, Kraichnan's (1976) analysis shows that $\nu(k|k_c)$ exhibits a marked cusp at $k \to k_c$. This cusp is due to the strong local energy transfers across $k = k_c$.

More recently, RG theory has been applied to this problem. Rose (1977) used this method to analyse the mathematically similar problem of passive scalar convection. A more general method of *iterative averaging* has been shown to be equivalent to RG (McComb 1982) and to lead to a Heisenberg-type effective viscosity for the energy cascade (McComb and Shanmugasundaram 1983).

Previously our theory has been presented (McComb 1982) as an iterative time-averaging of the equations of motion. Connection with RG was made by (*a*) Fourier-transforming into ω space, and (*b*) invoking the Taylor hypothesis of frozen convection to take the analysis into k space. Formally, therefore, we regarded our calculation as giving the 'isotropic part' of an (in general) anisotropic eddy viscosity. We also noted (McComb and Shanmugasundaram 1983) that a strength of our calculation was that it—unlike conventional RG theory—did not introduce a triple moment of the low-k modes into the low-k equation in the process of eliminating high-k modes. Our objective in the present paper is, firstly, to show that *iterative averaging* may be employed directly in k space, with exactly the same assumptions as the conventional RG analysis, and yet achieve the better separation of the frequencies just referred to. Secondly, to present some calculations for the effective viscosity at $k \leqslant k_c$. We begin in the next section by introducing the basic equations and discussing the conventional RG approach to them.

2. The basic equations

Let us consider an incompressible fluid of molecular viscosity ν_0 occupying a cubical box of side L. At a later stage we shall take the limit $L \to \infty$ (which is required for rigorous isotropy) and summations will be replaced by integrals. If we let the velocity field be $U_\alpha(\boldsymbol{x}, t)$ then the Fourier components of this are defined by

$$U_\alpha(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} U_\alpha(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}). \tag{2.1}$$

The equation of motion may be written as

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) U_\alpha(\boldsymbol{k}, t) = \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta(\boldsymbol{j}, t) U_\gamma(\boldsymbol{k}-\boldsymbol{j}, t) \tag{2.2}$$

and the continuity equation becomes

$$k_\alpha U_\alpha(\boldsymbol{k}, t) = 0 \tag{2.3}$$

where the inertial-transfer operator $M_{\alpha\beta\gamma}(\boldsymbol{k})$ is defined by

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = (2\mathrm{i})^{-1}(k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})) \tag{2.4}$$

and

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\boldsymbol{k}|^{-2}. \tag{2.5}$$

For isotropic turbulence, the pair correlation of velocities takes the form

$$\left(\frac{L}{2\pi}\right)^3 \langle U_\alpha(\boldsymbol{k}, t) U_\beta(-\boldsymbol{k}, t')\rangle = D_{\alpha\beta}(\boldsymbol{k}) Q(k; t, t') \tag{2.6}$$

where $\langle\ \rangle$ means the average value, and the expression for the energy spectrum follows in the usual way:

$$E(k, t) = 4\pi k^2 Q(k; t, t). \tag{2.7}$$

Let us now apply RG theory to the velocity field in much the same way as Rose (1977) applied it to passive scalar convection. We begin by taking our cut-off wavenumber to be $k_c = k_0$, where k_0 is large and, in general, of the order of the Kolmogorov dissipation wavenumber $k_d = (\varepsilon/\nu_0^3)^{1/4}$. We divide the velocity field at $k = k_0$ into

$$\begin{aligned} U_\alpha(\boldsymbol{k}, t) &= U_\alpha^<(\boldsymbol{k}, t) \qquad k \leqslant k_0 \\ &= U_\alpha^>(\boldsymbol{k}, t) \qquad k \geqslant k_0. \end{aligned} \tag{2.8}$$

The procedure then involves two stages.

(*a*) Eliminate the high-k modes by solving the equation for $U_\alpha^>$ (obtained from (2.2)) and substituting the solution in the equation for $U_\alpha^<$. Average over the high-k modes.

(*b*) Rescale $\boldsymbol{k}$, t and $U_\alpha^<$ so that the new equation looks like the original Navier–Stokes equation. This step involves the introduction of a renormalised eddy viscosity.

These two steps are then repeated for $k_c = k_1 < k_0$, and so on, until the scaled eddy viscosity reaches a fixed point.

The main problem faced when trying to implement this programme is how to tackle terms which couple $U_\alpha^<$ and $U_\alpha^>$. The advantage of the RG approach is that we can make some approximations. For example, the approximations made by Rose (1977) seem perfectly reasonable. They are as follows.

(i) $U_\alpha^<$ and $U_\alpha^>$ can be treated as statistically independent.

(ii) In any realisation, the $U_\alpha^>$ evolve much faster than the $U_\alpha^<$.

(iii) $U_\alpha^>$ is much smaller than $U_\alpha^<$ and the second order of small quantities may be neglected.

However, as we pointed out earlier, even with these approximations the equation for the low-k modes received a contribution from the high-k modes which includes the triple moment $\langle U^< U^< U^<\rangle$, as well as the required increment to the effective viscosity. In the next section we shall show that this does not happen with iterative averaging, even when implemented directly in k space and with the approximations listed above.

3. Iterative averaging and RG

Let us again divide up the velocity field at $k = k_0$, but this time we use a different notation to help distinguish our own approach from the RG analysis discussed in the previous section.

We put

$$\begin{aligned} U_\alpha(\boldsymbol{k}, t) &= U_\alpha^-(\boldsymbol{k}, t) \qquad k \leqslant k_0 \\ &= U_\alpha^+(\boldsymbol{k}, t) \qquad k \geqslant k_0. \end{aligned} \tag{3.1}$$

The averaging process is taken to be an ensemble average over the small scales such that

$$\begin{aligned} &\langle U_\alpha(\boldsymbol{k}, t)\rangle = U_\alpha^-(\boldsymbol{k}, t) \\ &\langle U_\alpha^-(\boldsymbol{k}, t)\rangle = U_\alpha^-(\boldsymbol{k}, t) \qquad \langle U_\alpha^+(\boldsymbol{k}, t)\rangle = 0. \end{aligned} \tag{3.2}$$

Substituting (3.1) into (2.2) and averaging according to (3.2) we obtain the low-frequency equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) U_\alpha^-(\boldsymbol{k}, t) - \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})\langle U_\beta^+(\boldsymbol{k}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle \\ &\qquad = \sum_{\boldsymbol{k}} M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta^-(\boldsymbol{j}, t) U_\beta^-(\boldsymbol{j}, t) U_\gamma^-(\boldsymbol{k}-\boldsymbol{j}, t) \qquad k_0 \leqslant \boldsymbol{j}, |\boldsymbol{k}-\boldsymbol{j}| \leqslant \infty. \end{aligned} \tag{3.3}$$

This is now our basic equation and our main objective is to eliminate the explicit terms with wavenumbers greater than k_0, which already only appear in a statistical sense through $\langle U^+ U^+\rangle$. We form the high-frequency equation by substituting (3.3) back into equation (2.2), with the result:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) U_\alpha^+(\boldsymbol{k}, t) \\ &\qquad = \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k})\{2U_\beta^-(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t) + U_\beta^+(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t) \\ &\qquad - \langle U_\beta^+(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle\}. \end{aligned} \tag{3.4}$$

With the approximations listed in the previous section, the solution to (3.4) may now be written as

$$U_\alpha^+(\boldsymbol{k}, t) = 2(\nu_0 k^2)^{-1} \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta^-(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t). \tag{3.5}$$

We now wish to obtain an expression for $\langle U_\beta^+(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle$ which occurs in the RHS of equation (3.3). Multiplying both sides of (3.5) by $U_\alpha^+(\boldsymbol{k}', t)$ and averaging, we then re-label as appropriate and remove dummy variables to avoid confusion to obtain

$$\langle U_\beta^+(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle = 2(\nu_0 j^2)^{-1} \sum_{\boldsymbol{p}} M_{\beta\rho\delta}(\boldsymbol{j}) U_\rho^-(\boldsymbol{p}, t)\langle U_\delta^+(\boldsymbol{j}-\boldsymbol{p}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle. \tag{3.6}$$

Now, from (2.6), with an obvious extension of the notation, we have

$$\langle U_\delta^+(\boldsymbol{j}-\boldsymbol{p}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle = \left(\frac{2\pi}{L}\right)^3 D_{\delta\gamma}(\boldsymbol{k}-\boldsymbol{j}) Q_0^+(|\boldsymbol{k}-\boldsymbol{j}|, t)\delta_{\boldsymbol{kp}} \tag{3.7}$$

where homogeneity implies

$$\boldsymbol{j}-\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{j} = 0.$$

Therefore

$$\boldsymbol{k} = \boldsymbol{p}.$$

Substituting (3.7) into (3.6) and summing over $\boldsymbol{p}$ to eliminate $\delta_{\boldsymbol{kp}}$ we obtain

$$\langle U_\beta^+(\boldsymbol{j}, t) U_\gamma^+(\boldsymbol{k}-\boldsymbol{j}, t)\rangle_0$$

$$= 2(\nu_0 j^2)^{-1}\left(\frac{2\pi}{L}\right)^3 M_{\beta\rho\delta}(\boldsymbol{j}) D_{\delta\gamma}(\boldsymbol{k}-\boldsymbol{j}) Q_0^+(|\boldsymbol{k}-\boldsymbol{j}|, t) U_\rho^-(\boldsymbol{k}, t). \tag{3.8}$$

Finally we substitute (3.8) into the RHS of equation (3.3). In the process we take two steps. Thus:

(i) $$M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\rho^-(\boldsymbol{k}, t) = M_{\rho\beta\gamma}(\boldsymbol{k}) U_\alpha^-(\boldsymbol{k}, t)$$

(ii) $$(2\pi/L)^3 \sum_j \to \int \mathrm{d}^3\boldsymbol{j} \qquad \text{as } L \to \infty$$

with the result

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2 + \delta\nu_0(k, t) k^2\right) U_\alpha^-(\boldsymbol{k}, t)$$

$$= \sum_j M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta^-(\boldsymbol{j}, t) U_\gamma^-(\boldsymbol{k}-\boldsymbol{j}, t) \qquad 0 \leqslant k \leqslant k_0 \tag{3.9}$$

where

$$\delta\nu_0(k, t) = \int \mathrm{d}^3\boldsymbol{j} \frac{L_{kj} Q_0^+(|\boldsymbol{k}-\boldsymbol{j}|, t)}{\nu_0 j^2 k^2} \tag{3.10}$$

for $k_0 \leqslant j$, $|\boldsymbol{k}-\boldsymbol{j}| \leqslant \infty$ and $0 \leqslant k \leqslant k_0$ and

$$L_{kj} = -2 M_{\rho\beta\gamma}(\boldsymbol{k}) M_{\beta\rho\delta}(\boldsymbol{j}) D_{\delta\gamma}(\boldsymbol{k}-\boldsymbol{j}). \tag{3.11}$$

This procedure is then repeated for successive wavenumbers $k_1, k_2, \ldots, k_n, \ldots$, such that $k_0 > k_1 > k_2 \ldots > k_n > \ldots$. By induction, the iteration for k_n yields

$$\left(\frac{\partial}{\partial t} + \nu_n(k, t) k^2 + \delta\nu_n(k, t) k^2\right) U_\alpha^-(\boldsymbol{k}, t)$$

$$= \sum_j M_{\alpha\beta\gamma}(\boldsymbol{k}) U_\beta^-(\boldsymbol{j}, t) U_\gamma^-(\boldsymbol{k}-\boldsymbol{j}, t) \qquad 0 \leqslant k \leqslant k_n \tag{3.12}$$

where the effective viscosity satisfies

$$\nu_{n+1}(k, t) = \nu_n(k, t) + \delta\nu_n(k, t) \tag{3.13}$$

and

$$\delta\nu_n(k, t) = \int \mathrm{d}^3\boldsymbol{j} \frac{L_{kj} Q_n^+(|\boldsymbol{k}-\boldsymbol{j}|, t)}{\nu_n(j, t) j^2 k^2} \tag{3.14}$$

for $k_n \leqslant j$, $|\boldsymbol{k}-\boldsymbol{j}| \leqslant k_{n-1}$ and $0 \leqslant k \leqslant k_n$.

4. RG calculation of the effective viscosity

Calculations have been carried out for the stationary case where the spectrum may be taken as the Kolmogorov form (1.1). The wavenumber bands are arbitrarily chosen as

$$k_n = h^n k_0 \qquad 0 \leqslant h \leqslant 1 \tag{4.1}$$

and we make the scaling transformation

$$k \to k_n \tilde{k}. \tag{4.2}$$

The scaled effective viscosity ν_n^* is defined by

$$\nu_n(k_n\tilde{k}) = \alpha^{1/2}\varepsilon^{1/3}k_n^{-4/3}\nu_n^*(\tilde{k}) \tag{4.3}$$

and equations (3.13) and (3.14) become

$$\nu_{n+1}^*(\tilde{k}) = h^{4/3}\{\nu_n^*(h\tilde{k}) + \delta\nu_n^*(h\tilde{k})\} \tag{4.4}$$

and

$$\delta\nu_n^*(\tilde{k}) = \frac{1}{4\pi}\int \mathrm{d}^3\tilde{j}\,\frac{L_{\tilde{k}\tilde{j}}|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|^{-11/3}}{\nu_n^*(\tilde{j})\tilde{k}^2\tilde{j}^2} \tag{4.5}$$

and $1 \leqslant \tilde{j}$, $|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}| \leqslant h^{-1}$.

Extensive computations of equations (4.3)-(4.5) have been carried out in connection with the formulation of the large eddy simulation equations for isotropic turbulence, and these will be reported elsewhere (McComb and Shanmugasundaram 1985). We shall note the general result that the recursion relation was found to reach a fixed point $\nu_{n+1}^*(\tilde{k}) = \nu_n^*(\tilde{k}) \equiv \nu_N^*(\tilde{k})$ and that this fixed point was independent of the RG parameters h and ν_0^*, and confine ourselves here to one example of the behaviour of $\nu_n^*(\tilde{k})$.

In figure 1, we show $\nu_n^*(\tilde{k})$ plotted against $\tilde{k}$ for various values of n and $\tilde{k}$ in the range $10^{-2} < \tilde{k} < 10^2$. Although we are concerned with $\nu_n^*(\tilde{k})$ for $\tilde{k} \leqslant 1$ (particularly when $n = N$), the transformation (4.4) shifts the effective viscosity from $\tilde{k} < 1$ to $\tilde{k} > 1$. On each cycle n, we need to know ν_n^* for $\tilde{k} > 1$ in order to calculate ν_{n+1}^* for $\tilde{k} < 1$. From figure 1, the evolution of ν_n^* with n for $\tilde{k} > 1$ is seen to settle down to the expected

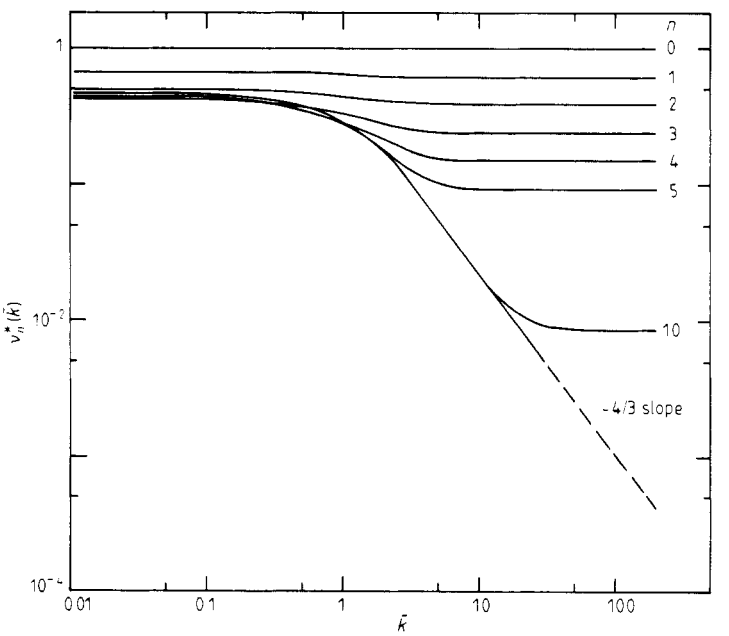

Figure 1. Evolution of the scaled eddy viscosity for all $\tilde{k}$; $h = 0.7$.

$\tilde{k}^{-4/3}$ law. This has been previously reported (McComb and Shanmugasundaram 1983) but in that reference the limits on the $\tilde{j}$ integration were incorrectly given as $1 < \tilde{j} < \infty$, which led to a numerical error. For example, we calculated the Kolmogorov constant to be $\alpha \sim 5.57$. With the correct wavenumber limits this result becomes $\alpha = 1.8$.

In figure 2, we show the behaviour of $\nu_n^*(\tilde{k})$ for $\tilde{k} \leqslant 1$ in more detail. Clearly as n increases, ν_n^* reaches a fixed point (constant value) for $\tilde{k} \ll 1$. As Kraichnan (1976) has pointed out, it is only when the high and low modes are so widely separated that the analogy with molecular viscosity becomes good. In this range we agree quite well with Kraichnan's result. However, as $\tilde{k} \to 1$, we would argue that the gentle roll-off seen in figure 2 (when compared with the cusp found with renormalised perturbation theory) suggests that RG is better at handling the strong local energy transfers across $\tilde{k} = 1$, although it should be noted that the two methods use different definitions of the effective viscosity.

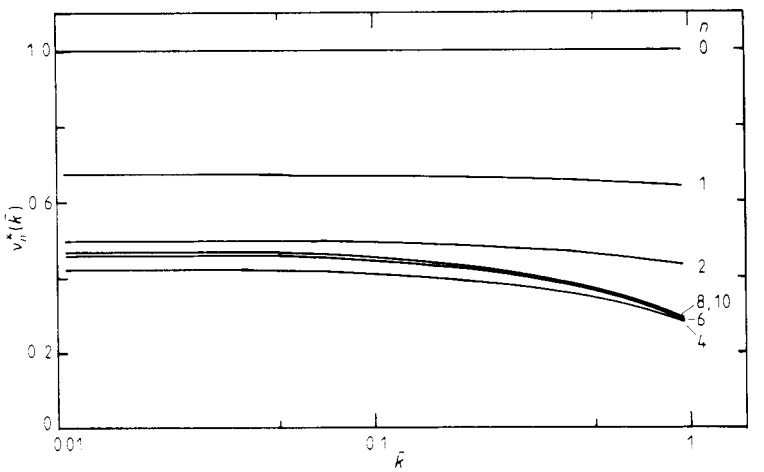

Figure 2. Evolution of the scaled eddy viscosity for $\tilde{k} \leqslant 1.0$; $h = 0.7$.

5. Conclusions

In view of the results presented, it seems that iterative averaging is a promising method of applying RG theory to the statistical description of well developed turbulence. There seem to be three key points in the method.

(i) In formulating the low-frequency equation, the high frequencies only appear in an average sense, through the variance: see equation (3.3).

(ii) When the variance of the high-frequency modes is calculated, the labelling k vector is found to be restricted to $0 \leqslant k \leqslant k_n$): see equations (3.8) and (3.9). Thus we have correctly identified the contribution to the eddy viscosity in the low-frequency equation.

(iii) No spurious low-frequency terms are introduced into the low-frequency equation in the process of eliminating the high-frequency modes: again, see equations (3.8) and (3.9).

A study of the underlying approximations can be made in conjunction with a numerical simulation, and work has begun on this problem.

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References

Batchelor G K 1971 *The theory of homogeneous turbulence* 2nd edn (Cambridge: CUP)
Heisenberg W 1948 *Z. Phys.* **124** 628
Kraichnan R H 1959 *J. Fluid Mech.* **5** 497
—— 1976 *J. Atmos. Sci.* **33** 1521
—— 1982 *Phys. Rev.* A **25** 3281
Kraichnan R H and Herring J R 1978 *J. Fluid Mech.* **88** 355
McComb W D 1978 *J. Phys. A: Math. Gen.* **11** 613
—— 1982 *Phys. Rev.* A **26** 1078
McComb W D and Shanmugasundaram V 1983 *Phys. Rev.* A **28** 2588
—— 1984 *J. Fluid Mech.* **143** 95
—— 1985 *J. Fluid Mech.* submitted
Rogallo R S and Moin P 1984 *Ann. Rev. Fluid Mech.* **16** 99
Rose H A 1977 *J. Fluid Mech.* **81** 719
Voke P R and Collins M W 1983 *PCH Physicochem. Hydrodyn.* **4** 119
Wilson K G 1975 *Rev. Mod. Phys.* **47** 773